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Dirac Spinors in an Inhomogeneous Cosmological Model

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Solutions of the Dirac equation in the Senovilla inhomogeneous cosmological model, which is free of a big-bang singularity, are presented. It is found that the energy density of the spinors is initially zero, but attains a maximum at some time and then decreases.

Recently Senovilla (1990) proposed a new class of inhomogeneous exact solutions to Einstein's field equations with a perfect-fluid source obeying a radiation-like equation of state. This solution is free from a big-bang singularity, contrary to standard models of cosmology. In the past, several improved methods have been proposed to remove the singularity, for example, by using nonminimal coupling with a scalar field (Sathyaprakash *et al.*, 1984, 1986).

The absence of an initial singularity in the Senovilla model is physically very interesting. However, these solutions are inhomogeneous, contrary to the standard model. Thus these solutions need examination under various physical conditions so as to assess their relevance in the context of the observable universe.

As a first step, one would like to study the behavior of an elementary particle in this inhomogeneous model. Since an important class of elementary particles have spin $\frac{1}{2}$ and are represented by Dirac spinors, it is desirable to study their properties in this new metric. It may be noted that Senovilla's metric has some similarity with a special case of the metric studied by Kamran and McLenaghan (1984) in the context of the Dirac equation.

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The line element of the Senovilla metric has the form

$$ds^2 = e^{2f}(-dt^2 + dx^2) + G(q\, dy^2 + q^{-1}\, dz^2) \tag{1}$$

where

$$e^f = \cosh^2(at)\cosh(3ax) \tag{2a}$$

$$G = \cosh(at)\sinh(3ax)\cosh^{-2/3}(3ax) \tag{2b}$$

$$q = \cosh^3(at)\sinh(3ax) \tag{2c}$$

Here the range of the coordinates is

$$-\infty < t, x, y, z < \infty \tag{3}$$

and a is an arbitrary constant (Senovilla, 1990).

Now the action for Dirac spinors ψ is taken as

$$A = \frac{i}{2}\int d^4x\, (-g)^{1/2}\bar{\psi}\gamma^\mu D_\mu \psi \tag{4}$$

where γ^μ are Dirac matrices in curved space and D_μ is the covariant derivative given by

$$D_\mu = \partial_\mu - \Gamma_\mu \tag{5}$$

where

$$\Gamma_\mu = \tfrac{1}{4}(\partial_\mu h^\rho_a + \{^{\rho}_{\sigma\mu}\} h^\sigma_a) g_{\nu\rho} h^\nu_b \tilde{\gamma}^a \tilde{\gamma}^b \tag{6}$$

Here $\{^{\rho}_{\sigma\mu}\}$ is the affine connection and h^ν_b are tetrads defined as (Srivastava, 1989)

$$h^\mu_a h^\nu_b g_{\mu\nu} = \eta_{ab} \tag{7}$$

η_{ab} is the metric for flat space-time, and the $g_{\mu\nu}$ are given by the line element [see equation (1)].

$\tilde{\gamma}^a$ are i-times the standard Dirac matrices in flat space-time, which are related to γ^μ by

$$\gamma^\mu = h^\mu_a \tilde{\gamma}^a \tag{8}$$

These matrices satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{9a}$$

$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = 2\eta^{ab} \tag{9b}$$

Note that the action given in equation (4) is conformally invariant. Under conformal transformation

$$g_{\mu\nu} \rightarrow e^{2f} g_{\mu\nu}^{(c)} \tag{10}$$

The action for ψ can be rewritten as

$$A = \frac{i}{2} \int d^4x \, (-g^{(c)}) \bar{\psi}^{(c)} \gamma^{\mu(c)} D_{\mu}^{(c)} \psi^{(c)} \tag{11}$$

where

$$\gamma^{\mu(c)} = e^{f} \gamma^{\mu}$$

$$\psi^{(c)} = e^{3f/2} \psi$$

$$D_{\mu}^{(c)} = \partial_{\mu} - \Gamma_{\mu}^{(c)}$$

The Dirac equation obtained from the action in equation (11) has the form

$$i\gamma^{\mu(c)}(\partial_{\mu} - \Gamma_{\mu}^{(c)}) \psi^{(c)} = 0 \tag{12}$$

Under the conformal transformation [see equation (10)]

$$ds^2 = e^{2f} \, ds^{2(c)} \tag{13}$$

with

$$ds^{2(c)} = -dt^2 + dx^2 + B \, dy^2 + C \, dz^2 \tag{14}$$

where

$$B = \tanh^2(3ax) \operatorname{sech}^{2/3}(3ax)$$

$$C = \operatorname{sech}^6(at) \operatorname{sech}^{8/3}(3ax)$$

Using the metric given by the line element (14), we find that the Dirac equation takes the form

$$\begin{aligned}
&\tilde{\gamma}^0 \partial_0 \psi^{(c)} + \tilde{\gamma}' \left[\partial_1 - \frac{a}{4} \frac{3 - \sinh^2(3ax)}{\cosh^{8/3}(3ax)} \tanh(3ax) \right. \\
&\left. - 2a \tanh(3ax) + \frac{a}{4} \frac{3 - \sinh^2(3ax)}{\cosh^2(3ax) \tanh(3ax)} \right] \psi^{(c)} \\
&+ \frac{\cosh^{1/3}(3ax)}{\tanh(3ax)} \tilde{\gamma}^2 \partial_2 \psi^{(c)} \\
&+ \cosh^{4/3}(3ax) \cosh^3(at) \tilde{\gamma}^3 \partial_3 \psi^{(c)} = 0
\end{aligned} \tag{15}$$

Writing

$$\psi^{(c)}=\begin{pmatrix}\psi_1^{(c)}\\ \psi_2^{(c)}\\ \psi_3^{(c)}\\ \psi_4^{(c)}\end{pmatrix}=\frac{\cosh^{13/12}(3ax)}{(2\pi)\sinh^{1/3}(3ax)}$$

$$\times\exp[-\tfrac{1}{8}\cosh^{-8/3}(3ax)+\tfrac{3}{2}\cosh^{-2/3}(3ax)]$$

$$\times e^{ik_2y+ik_3z}\begin{pmatrix}u_1\\ u_2\\ u_3\\ u_4\end{pmatrix} \tag{16}$$

(k_2, k_3 are components of wave vectors in the y and z directions, respectively), we find that equation (15) reduces to four coupled equations

$$\partial_0 u_1+\left(\partial_1+k_2\frac{\cosh^{4/3}(3ax)}{\sinh(3ax)}\right)u_4$$
$$+ik_3\cosh^{4/3}(3ax)\cosh^3(at)u_3=0 \tag{17a}$$

$$\partial_0 u_2+\left(\partial_1-k_2\frac{\cosh^{4/3}(3ax)}{\sinh(3ax)}\right)u_3$$
$$-ik_3\cosh^{4/3}(3ax)\cosh^3(at)u_4=0 \tag{17b}$$

$$\partial_0 u_3+\left(\partial_1+k_2\frac{\cosh^{4/3}(3ax)}{\sinh(3ax)}\right)u_2$$
$$+ik_3\cosh^{4/3}(3ax)\cosh^3(at)u_1=0 \tag{17c}$$

$$\partial_0 u_4+\left(\partial_1-k_2\frac{\cosh^{4/3}(3ax)}{\sinh(3ax)}\right)u_1$$
$$-ik_3\cosh^{4/3}(3ax)\cosh^3(at)=0 \tag{17d}$$

The solutions of equations (17) turn out to be column matrices,

$$u_1=\begin{pmatrix}(-ik_3/3a)\sinh(at)[3+\sinh^2(at)]\\ 0\\ \cosh^{-4/3}(3ax)\\ \exp\{k_2\int[\cosh^{4/3}(3ax)]/[\sinh(3ax)]\,dx\}\\ 0\end{pmatrix} \tag{18a}$$

$$u_2=\begin{pmatrix}(ik_3/3a)\sinh(at)[3+\sinh^2(at)]\\ \exp\{-k_2\int[\cosh^{4/3}(3ax)]/[\sinh(3ax)]\,dx\}\\ \cosh^{-4/3}(3ax)\end{pmatrix} \tag{18b}$$

$$u_3 = \begin{pmatrix} \cosh^{-4/3}(3ax) \\ \exp\{k_2 \int [\cosh^{4/3}(3ax)]/[\sinh(3ax)]\, dx\} \\ (-ik_3/3a)\sinh(at)[3+\sinh^2(at)] \\ 0 \end{pmatrix} \tag{18c}$$

$$u_4 = \begin{pmatrix} \exp\{-k_2 \int [\cosh^{4/3}(3ax)]/[\sinh(3ax)]\, dx \\ \cosh^{-4/3}(3ax) \\ 0 \\ (ik_3/3a)\sinh(at)[3+\sinh^2(at)] \end{pmatrix} \tag{18d}$$

Accordingly the components of the Dirac ψ are

$$\begin{aligned} \psi_1 &= N_1 F(x, t) u_1\, e^{ik_2 y + ik_3 z} \\ \psi_2 &= N_2 F(x, t) u_2\, e^{ik_2 y + ik_3 z} \\ \psi_3 &= N_3 F(x, t) u_3\, e^{ik_2 y + ik_3 z} \\ \psi_4 &= N_4 F(x, t) u_4\, e^{ik_2 y + ik_3 z} \end{aligned} \tag{19}$$

where N_1, N_2, N_3, N_4 are normalization constants and

$$F(x, t) = \frac{\cosh^{13/12}(3ax)}{\sinh^{1/3}(3ax)} \times \frac{\exp[-\frac{1}{8}\cosh^{-8/3}(3ax) + \frac{3}{2}\cosh^{-2/3}(3ax)]}{\cosh^{3/2}(3ax)\cosh^3(at)} \tag{20}$$

Note that the normalization constants are evaluated from the expression

$$\int_{x,t=\text{const}} dy\, dz\, \psi_{ik}^\dagger \tilde{\gamma}^0 \psi_{ik'} = (2\pi)^2 \delta(k_2 - k_2')\delta(k_3 - k_3') \tag{21}$$

The energy-momentum tensor for the Dirac field is given by

$$T_{\mu\nu} = \frac{i}{2}(\bar{\psi}\gamma_\mu D_\nu \psi) \tag{22}$$

Since the action does not involve any self-interaction of the ψ field, the energy-momentum tensor for the perfect fluid can be used. This is given as

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \tag{23}$$

where ρ is the energy density, p is the average pressure, and the four-velocity

$$u^\mu = (e^{-f}, 0, 0, 0) \tag{24}$$

is normalized as $u^\mu u_\mu = -1$.

Since ψ is conformally invariant, at the classical level we have the relation

$$T^\mu_\mu = 0 \tag{25a}$$

which means that

$$T_0^0 + T_1^1 + T_2^2 + T_3^3 = 0 \tag{25b}$$

Now using equations (23)-(25), we have

$$\rho = T_1^1 + T_2^2 + T_3^3 \tag{26}$$

which with the help of equation (22) becomes

$$\begin{aligned}\rho = {} & \frac{k_2 k_3}{(2\pi)^2 3a} \frac{\sinh(at)[3+\sinh^2(at)]}{\cosh^8(at)\cosh^4(3ax)} \cosh^{13/6}(3ax) \\ & \times \exp[-\tfrac{1}{4}\cosh^{-8/3}(3ax) + 3\cosh^{-2/3}(3ax)] \\ & \times \left\{ (N_1N_3 + N_2N_4) \frac{\cosh^{4/3}(3ax)}{\sinh(3ax)} \right. \\ & \times \left[\exp\left(k_2 \int \frac{\cosh^{4/3}(3ax)}{\operatorname{sonh}(3ax)}\, dx \right) - \exp\left(-k_2 \int \frac{\cosh^{4/3}(3ax)}{\sinh(3ax)}\, dx \right) \right] \\ & + 2(N_1N_4 + N_2N_3) \left[\exp\left(k_2 \int \frac{\cosh^{4/3}(3ax)}{\sinh(3ax)}\, dx \right) \right. \\ & \left. \left. + \exp\left(-k_2 \int \frac{\cosh^{4/3}(3ax)}{\sinh(3ax)}\, dx \right) \right] \right\}\end{aligned} \tag{27}$$

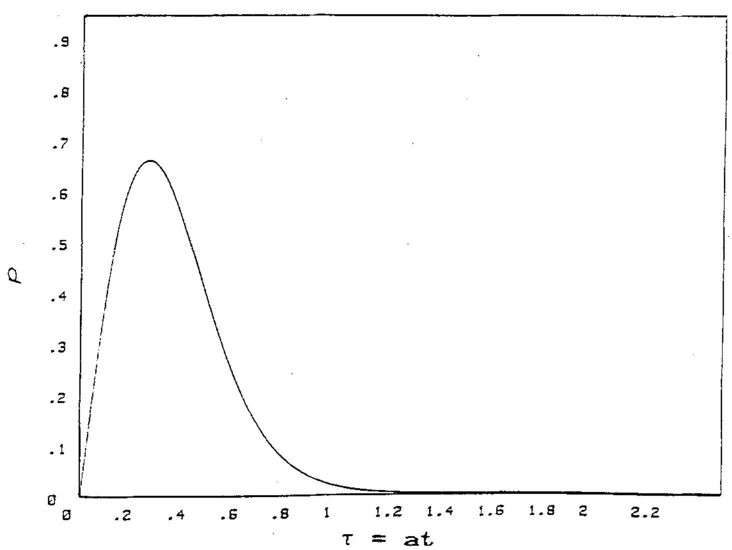

Fig. 1. A plot of ρ against $\tau = at$, keeping x, k_2, and k_3 constant.

A plot of ρ (after appropriate scaling) is given in Figure 1 at $x = \text{const}$ hypersurface. It is seen that $\rho(t=0)$ is zero, but increases with time and attains a maximum value at $\tau = at = 0.39$. It rapidly falls after this time and becomes zero again for large τ. We have restricted the calculations to positive time only, since as for negative time [see equation (27)] ρ becomes negative which is unphysical.

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